

Wall-crossing made smooth

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ABSTRACT: In $D = 4, \mathcal{N} = 2$ theories on $\mathbb{R}^{3,1}$, the index receives contributions not only from single-particle BPS states, counted by the BPS indices, but also from multi-particle states made of BPS constituents. In a recent work [1], a general formula expressing the index in terms of the BPS indices was proposed, which is smooth across walls of marginal stability and reproduces the expected single-particle contributions. In this note, I analyze the two-particle contributions predicted by this formula, and show agreement with the spectral asymmetry of the continuum of scattering states in the supersymmetric quantum mechanics of two non-relativistic, mutually non-local dyons. This provides a physical justification for the error function profile used in the mathematics literature on indefinite theta series, and in the physics literature on black hole partition functions.

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1. Introduction

The recent work [1] proposed a general formula for the index $\mathcal{I}(R, u, C)$ in four-dimensional field theories on $\mathbb{R}^{3,1}$ with $\mathcal{N} = 2$ supersymmetry. This index can be understood as the partition function on \mathbb{R}^3 times an Euclidean circle of radius R , with periodic boundary conditions for fermions, chemical potentials C conjugate to the electromagnetic charge γ , and with an insertion of a suitable four-fermion vertex so as to saturate fermionic zero-modes. Equivalently, it can be defined as a trace

$$\mathcal{I}(R, u, C) = -\frac{1}{2} \text{Tr}_{\mathcal{H}(u)} (-1)^{2J_3} (2J_3)^2 \sigma_\gamma e^{-2\pi R H - 2\pi i \langle \gamma, C \rangle}, \quad (1.1)$$

over the full Hilbert space $\mathcal{H}(u)$ of the theory on \mathbb{R}^3 (here J_3 is the angular momentum operator around a fixed axis, and σ_γ is a charge-dependent sign, satisfying the quadratic refinement property $\sigma_\gamma \sigma_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \sigma_{\gamma + \gamma'}$).

Unlike the BPS indices $\Omega(\gamma, u)$, which count single-particle BPS states and exhibit discontinuities across walls of marginal stability, $\mathcal{I}(R, u, C)$ is a smooth function of the Coulomb branch moduli u , away from the loci where additional states become massless. This is possible because $\mathcal{I}(R, u, C)$ receives contributions not only from single-particle BPS states, but also from the continuum of multi-particle states. Indeed, while multi-particle states do not saturate the BPS bound $M = |Z_\gamma|$, the density of bosonic and fermionic states are not necessarily equal, as noted early on in [2, 3] (see [4, 5, 6] for recent discussions in the context of two-dimensional superconformal field theories). Still, only multi-particle states made of BPS constituents can contribute, so one expects that the index can be expressed in terms of the BPS indices $\Omega(\gamma, u)$.

In [1], using insight from the study of the hyperkähler metric on the Coulomb branch in the theory compactified down to three dimensions [7], and by analogy with a similar construction

in the context of the hypermultiplet moduli space in string vacua [8, 9], we proposed a general formula for the index¹

$$\mathcal{I}(R, u, C) = -\frac{R^2}{2} \Im(\bar{X}^\Lambda F_\Lambda) + \frac{R}{16i\pi^2} \sum_{\gamma} \Omega(\gamma, u) \int_{\ell_{\gamma}} \frac{dt}{t} (t^{-1} Z_{\gamma} - t \bar{Z}_{\gamma}) \log(1 - \mathcal{X}_{\gamma}(t)). \quad (1.2)$$

where ℓ_{γ} are the BPS rays $\{t \in \mathbb{C}^{\times} : Z_{\gamma}/t \in i\mathbb{R}^{-}\}$, and \mathcal{X}_{γ} are the solutions to the system of integral equations [7]

$$\mathcal{X}_{\gamma} = \mathcal{X}_{\gamma}^{\text{sf}} \exp \left[\sum_{\gamma'} \frac{\Omega(\gamma', u)}{4\pi i} \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t-t'} \log(1 - \mathcal{X}_{\gamma'}(t')) \right], \quad (1.3)$$

with $\mathcal{X}_{\gamma}^{\text{sf}}$ providing the ‘semi-flat’, large R approximation to \mathcal{X}_{γ} ,

$$\mathcal{X}_{\gamma}^{\text{sf}} = \sigma_{\gamma} e^{-\pi i R (t^{-1} Z_{\gamma} - t \bar{Z}_{\gamma}) - 2\pi i \langle \gamma, C \rangle}. \quad (1.4)$$

The \mathcal{X}_{γ} ’s are holomorphic functions on the twistor space \mathcal{Z} of the Coulomb branch $\mathcal{M}_3(R)$, which provide canonical Darboux coordinates for the holomorphic symplectic structure on \mathcal{Z} . They can also be understood as vevs of certain infrared line operators [11]. In the limit $R \rightarrow \infty$, a formal solution to the system (1.3) is obtained by substituting $\mathcal{X}_{\gamma} \rightarrow \mathcal{X}_{\gamma}^{\text{sf}}$ on the r.h.s. and iterating. This leads an expansion of the form

$$\mathcal{X}_{\gamma} = \mathcal{X}_{\gamma}^{\text{sf}} \exp \left[\sum_T \prod_{(i,j) \in T_1} \langle \alpha_i, \alpha_j \rangle \prod_{i \in T_0} \bar{\Omega}(\alpha_i, u) g_T \right] \quad (1.5)$$

where T runs over trees decorated by charges α_i such that $\gamma = \sum \alpha_i$, g_T are certain iterated contour integrals [7, 12], and $\bar{\Omega}(\gamma, u) = \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d, u)$ are the ‘rational BPS indices’ [13, 14, 15], which arise from expanding the log in (1.3). Substituting in (1.2), one obtains a formal expansion

$$\mathcal{I} = \mathcal{I}^{(0)} + \sum_{\gamma} \mathcal{I}_{\gamma}^{(1)} + \sum_{\gamma, \gamma'} \mathcal{I}_{\gamma, \gamma'}^{(2)} + \dots \quad (1.6)$$

where $\mathcal{I}^{(0)}$ stands for the first term in (1.2), while $\mathcal{I}_{\gamma_1, \dots, \gamma_n}^{(n)}$, proportional to $\bar{\Omega}(\gamma_1, u) \dots \bar{\Omega}(\gamma_n, u)$ is interpreted as the contribution of a multi-particle state of charge $\{\gamma_1, \dots, \gamma_n\}$ to the index. In particular, the one-particle contribution is obtained by replacing $\mathcal{X}_{\gamma} \rightarrow \mathcal{X}_{\gamma}^{\text{sf}}$ in (1.2), leading to

$$\mathcal{I}_{\gamma}^{(1)} = \frac{R}{4\pi^2} \sigma_{\gamma} \bar{\Omega}(\gamma, u) |Z_{\gamma}| K_1(2\pi R |Z_{\gamma}|) e^{-2\pi i \langle \gamma, C \rangle} \quad (1.7)$$

In [1], we matched this result with the index of a relativistic particle of charge γ and mass $|Z_{\gamma}|$. To define the index, we regulated the infrared divergences by switching on a chemical potential θ for the rotations J_3 in the xy plane, restricting the z direction to a finite interval of length L , and removing the regulators as follows,

$$\mathcal{I}_{\gamma}^{(1)} = 2R \lim_{\substack{\theta \rightarrow 2\pi \\ L \rightarrow \infty}} \partial_{\theta}^2 \left[\frac{\sin^2(\theta/2)}{\pi L} \text{Tr}_{\mathcal{H}_1(u)} (\sigma_{\gamma} e^{-2\pi R H + i\theta J_3 - 2\pi i \langle \gamma, C \rangle}) \right]. \quad (1.8)$$

where $\mathcal{H}_1(u)$ is the one-particle Hilbert space. The same regulator should then be used to define the full index (1.1). Our aim in this note is to perform a similar check for the two-particle contribution.

¹Various refinements of this index have been introduced in [10], but lie beyond the scope of this note.

2. Two-particle contribution to the index

According to the conjecture (1.2), the contribution of a two-particle state with charges $\{\gamma, \gamma'\}$ to the index is obtained by inserting the one-particle approximation to (1.3) in (1.2),

$$\mathcal{I}_{\gamma, \gamma'}^{(2)} = -\frac{R}{64\pi^3} \sum_{\gamma, \gamma'} \langle \gamma, \gamma' \rangle \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \int_{\ell_\gamma} \frac{dt}{t} \int_{\ell_{\gamma'}} \frac{dt'}{t'} \frac{t+t'}{t-t'} (t^{-1} Z_\gamma - t \bar{Z}_\gamma) \mathcal{X}_\gamma^{\text{sf}}(t) \mathcal{X}_{\gamma'}^{\text{sf}}(t') + (\gamma \leftrightarrow \gamma') \quad (2.1)$$

Defining $\psi_\gamma = \arg Z_\gamma$, $\psi_{\gamma\gamma'} = \psi_\gamma - \psi_{\gamma'}$ and changing the integration variables to $t = ie^{i\psi_\gamma} e^{x_+ + x_-}$, $t' = ie^{i\psi_{\gamma'}} e^{x_+ - x_-}$, this can be rewritten as

$$\begin{aligned} \mathcal{I}_{\gamma, \gamma'}^{(2)} = & \frac{iR}{16\pi^3} (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \sigma_{\gamma+\gamma'} \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dx_- \coth \left(x_- + \frac{i}{2} \psi_{\gamma\gamma'} \right) \\ & \times [|Z_\gamma| \cosh(x_+ + x_-) + |Z_{\gamma'}| \cosh(x_+ - x_-)] e^{-2\pi R |Z_\gamma| \cosh(x_+ + x_-) - 2\pi R |Z_{\gamma'}| \cosh(x_+ - x_-) - 2\pi i \langle \gamma + \gamma', C \rangle}. \end{aligned} \quad (2.2)$$

We shall be interested in the behavior of $\mathcal{I}_{\gamma, \gamma'}^{(2)}$ in the vicinity of the wall of marginal stability where $\psi_{\gamma, \gamma'} \rightarrow 0$, in the limit $R \rightarrow \infty$. We assume that γ and γ' are primitive vectors generating the positive cone of BPS states whose central charges align at the wall, so that $\bar{\Omega}(\gamma, u)$ and $\bar{\Omega}(\gamma', u)$ are constant across the wall, and equal to $\Omega(\gamma, u)$ and $\Omega(\gamma', u)$. Away from the wall, the integrals over x_+ and x_- are dominated by saddle points at $x_+ = x_- = 0$, producing²

$$\mathcal{I}_{\gamma, \gamma'}^{(2)} \approx \frac{1}{32\pi^3} (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \sigma_{\gamma+\gamma'} \frac{|Z_\gamma| + |Z_{\gamma'}|}{|Z_\gamma| |Z_{\gamma'}|} \cot \left(\frac{1}{2} \psi_{\gamma\gamma'} \right) e^{-2\pi R (|Z_\gamma| + |Z_{\gamma'}|) - 2\pi i \langle \gamma + \gamma', C \rangle}. \quad (2.3)$$

However, the saddle point approximation breaks down near the wall where $\psi_{\gamma\gamma'} \rightarrow 0$, as the saddle point collides with the pole at $x_- = -\frac{i}{2} \psi_{\gamma\gamma'}$. To deal with this, we first perform the integral over x_+ , which is dominated by a saddle point at

$$x_+ \sim \frac{|Z_{\gamma'}| - |Z_\gamma|}{|Z_\gamma| + |Z_{\gamma'}|} x_- + \mathcal{O}(x_-^2). \quad (2.4)$$

In the limit $R \rightarrow \infty$, $\mathcal{I}_{\gamma, \gamma'}^{(2)}$ is well approximated by

$$\begin{aligned} \mathcal{I}_{\gamma, \gamma'}^{(2)} \approx & \frac{i}{16\pi^3} (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \sigma_{\gamma+\gamma'} \sqrt{R(|Z_\gamma| + |Z_{\gamma'}|)} \\ & \times \int_{-\infty}^{\infty} dx_- \coth \left(x_- + \frac{i}{2} \psi_{\gamma\gamma'} \right) e^{-2\pi R (|Z_\gamma| + |Z_{\gamma'}|) - \frac{4\pi R |Z_\gamma| |Z_{\gamma'}|}{|Z_\gamma| + |Z_{\gamma'}|} x_-^2 - 2\pi i \langle \gamma + \gamma', C \rangle}. \end{aligned} \quad (2.5)$$

In the limit $\psi_{\gamma\gamma'} \rightarrow 0$, we can further approximate $\coth(x) \sim 1/x$ and evaluate the integral using the formula [17, 4.18], valid for α and β real and non-zero,

$$\int_{-\infty}^{\infty} \frac{dz}{z - i\alpha} e^{-\beta^2 z^2} = i\pi \operatorname{sgn}(\alpha) e^{\alpha^2 \beta^2} \operatorname{Erfc}(|\alpha\beta|) \quad (2.6)$$

²The analysis in this section bears some similarities with the one in [16].

where Erfc is the complementary error function. Noting that

$$|Z_\gamma| + |Z_{\gamma'}| - |Z_{\gamma+\gamma'}| \sim \frac{1}{2} m_{\gamma\gamma'} \psi_{\gamma\gamma'}^2, \quad (2.7)$$

where $m_{\gamma\gamma'} = \frac{|Z_\gamma||Z_{\gamma'}|}{|Z_\gamma|+|Z_{\gamma'}|}$ is the reduced mass of the two-particle system, we find

$$\begin{aligned} \mathcal{I}_{\gamma,\gamma'}^{(2)} &\approx \frac{1}{16\pi^2} (-1)^{\langle\gamma,\gamma'\rangle} \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \langle\gamma,\gamma'\rangle \sigma_{\gamma+\gamma'} \sqrt{R(|Z_\gamma| + |Z_{\gamma'}|)} \\ &\times \text{sgn}(\psi_{\gamma\gamma'}) \text{Erfc}\left(|\psi_{\gamma\gamma'}| \sqrt{\pi R m_{\gamma\gamma'}}\right) e^{-2\pi R|Z_{\gamma+\gamma'}| - 2\pi i \langle\gamma+\gamma', C\rangle} \end{aligned} \quad (2.8)$$

The two-particle contribution is discontinuous across the wall: as $\psi_{\gamma\gamma'}$ goes from negative to positive, $\mathcal{I}_{\gamma,\gamma'}^{(2)}$ jumps by

$$\Delta \mathcal{I}_{\gamma,\gamma'}^{(2)} \approx \frac{1}{8\pi^2} (-1)^{\langle\gamma,\gamma'\rangle} \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \langle\gamma,\gamma'\rangle \sigma_{\gamma+\gamma'} \sqrt{R(|Z_\gamma| + |Z_{\gamma'}|)} e^{-2\pi R|Z_{\gamma+\gamma'}| - 2\pi i \langle\gamma+\gamma', C\rangle}. \quad (2.9)$$

On the other hand, the one-particle contribution $\mathcal{I}_{\gamma,\gamma'}^{(1)}$ is also discontinuous across the wall, due to the fact that the one-particle index $\bar{\Omega}(\gamma + \gamma')$ jumps [18]:³

$$\bar{\Omega}(\gamma + \gamma', u) = \bar{\Omega}^+(\gamma + \gamma') - (-1)^{\langle\gamma,\gamma'\rangle} |\langle\gamma,\gamma'\rangle| \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \Theta(\langle\gamma,\gamma'\rangle \psi_{\gamma\gamma'}). \quad (2.10)$$

The first term in (2.10) corresponds to the one-particle index on the side where $\langle\gamma,\gamma'\rangle \psi_{\gamma\gamma'} < 0$, so that the two states of charge γ and γ' cannot form a BPS bound state, while the second term is the contribution of the BPS bound state which exists on the side where $\langle\gamma,\gamma'\rangle \psi_{\gamma\gamma'} > 0$.

Inserting (2.10) in $\mathcal{I}_{\gamma+\gamma'}^{(1)}$, and taking the limit $R \rightarrow \infty$, we find

$$\begin{aligned} \mathcal{I}_{\gamma+\gamma'}^{(1)} &\approx \left[\bar{\Omega}^+(\gamma + \gamma') - (-1)^{\langle\gamma,\gamma'\rangle} |\langle\gamma,\gamma'\rangle| \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \Theta(\langle\gamma,\gamma'\rangle \psi_{\gamma\gamma'}) \right] \\ &\times \sigma_{\gamma+\gamma'} \frac{\sqrt{R|Z_{\gamma+\gamma'}|}}{8\pi^2} e^{-2\pi R|Z_{\gamma+\gamma'}| - 2\pi i \langle\gamma+\gamma', C\rangle}, \end{aligned} \quad (2.11)$$

whose jump exactly compensates (2.9). In fact, neglecting the difference between $|Z_{\gamma+\gamma'}|$ and $|Z_\gamma| + |Z_{\gamma'}|$ under the square root (which cannot be told apart in our approximation), the sum of (2.8) and (2.11) can be written as

$$\begin{aligned} \mathcal{I}_{\gamma+\gamma'}^{(1)} + \mathcal{I}_{\gamma,\gamma'}^{(2)} &\approx \left\{ \bar{\Omega}^+(\gamma + \gamma') \right. \\ &\quad \left. - \frac{1}{2} (-1)^{\langle\gamma,\gamma'\rangle} |\langle\gamma,\gamma'\rangle| \bar{\Omega}(\gamma) \bar{\Omega}(\gamma') \left(1 + \text{Erf}\left(\text{sgn}(\langle\gamma,\gamma'\rangle) \psi_{\gamma\gamma'} \sqrt{\pi R m_{\gamma\gamma'}}\right) \right) \right\} \\ &\times \sigma_{\gamma+\gamma'} \frac{\sqrt{R|Z_{\gamma+\gamma'}|}}{8\pi^2} e^{-2\pi R|Z_{\gamma+\gamma'}| - 2\pi i \langle\gamma+\gamma', C\rangle}, \end{aligned} \quad (2.12)$$

where we have used the identity $\text{Erf}(x) = \text{sgn}(x) (1 - \text{Erfc}(|x|))$. In plain words, the addition of the two-particle contribution to $\mathcal{I}_{\gamma+\gamma'}^{(1)}$ has converted the step function $\Theta(x)$ in (2.10) into the smooth function $\frac{1}{2}[1 + \text{Erf}(x)]$. This shows that the sum of the one and two-particle contributions is not only continuous, but also differentiable across the wall (see Figure 1 for illustration), which acquires a finite width of order $1/\sqrt{R m_{\gamma,\gamma'}}$ as a function of the relative phase $\psi_{\gamma\gamma'}$ between the central charges Z_γ and $Z_{\gamma'}$. It would be interesting to generalize this computation to the case of non-primitive wall-crossing, and to relax the non-relativistic limit $R \rightarrow \infty$.

³Here $\Theta(x)$ denotes the Heaviside step function, equal to 1 when $x > 0$ and 0 otherwise.

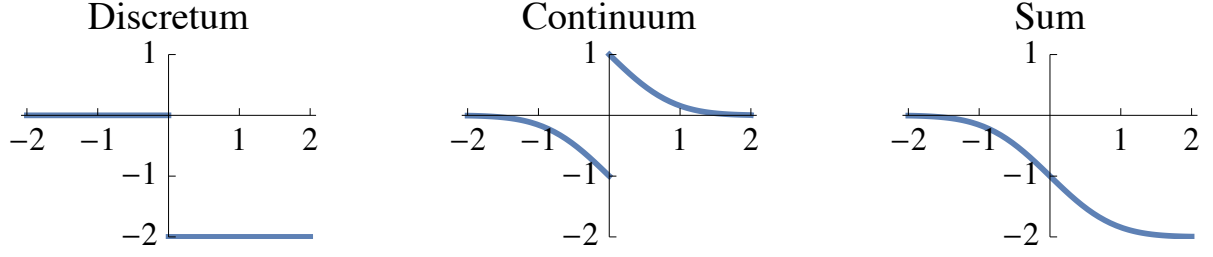


Figure 1: Behavior of the one-particle contribution $(-2\Theta(x))$, two-particle contribution $(\text{sign}(x) \text{Erfc}(|x|))$ and their sum $(-1 - \text{Erf}(x))$ to the index with total charge $\gamma + \gamma'$ across a wall where the phases of Z_γ and $Z_{\gamma'}$ align ($x \rightarrow 0$).

3. Supersymmetric electron-monopole quantum mechanics

Our goal in the remainder of this note is to derive the two-particle contribution (2.8) from the supersymmetric quantum mechanics of a system of two non-relativistic particles with mutually non-local primitive charges γ, γ' . After factoring out the center of mass degrees of freedom, which can be treated as in (1.8), and the internal degrees of freedom, counted by $\bar{\Omega}(\gamma)\bar{\Omega}(\gamma')$, the system is described by $\mathcal{N} = 4$ quantum mechanics with Hamiltonian [19, 20]⁴

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 - \frac{q}{2m} \vec{B} \cdot \vec{\sigma} \otimes (1_2 - \sigma_3) + \frac{1}{2m} \left(\vartheta - \frac{q}{r} \right)^2. \quad (3.1)$$

where $q = \frac{1}{2}\langle\gamma, \gamma'\rangle$ is half the Dirac-Schwinger-Zwanziger product of the electromagnetic charges, $\vec{B} = \frac{\vec{r}}{r^3}$ is the magnetic field of a unit charge magnetic monopole sitting at the origin, \vec{A} is the corresponding gauge potential, $\vec{\sigma}$ are the Pauli matrices, and $m = m_{\gamma\gamma'}$ is the reduced mass of the two-particle system. Classically, the system has bound states for $q\vartheta > 0$, no bound states for $q\vartheta < 0$, and a continuum of scattering states with energy $E \geq E_c = \frac{\vartheta^2}{2m}$. The parameter ϑ is fixed by equating E_c with the binding energy,

$$\frac{\vartheta^2}{2m} = |Z_\gamma| + |Z_{\gamma'}| - |Z_{\gamma+\gamma'}|, \quad (3.2)$$

so $\vartheta \sim m\psi_{\gamma\gamma'}$ near the wall, cf. (2.7). Quantum mechanically, H describes two bosonic degrees of freedom with helicity $h = 0$, and one fermionic doublet with helicity $h = \pm 1/2$ and gyromagnetic ratio $g = 4$. This unusual value is fixed by the requirement of supersymmetry, and can be understood as the combined effect of electromagnetic and scalar interactions [22]. Indeed, the Hamiltonian (3.1) commutes with the four supercharges (here $\vec{\Pi} = \vec{p} - q\vec{A}$) [19, 20, 21]

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -i\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma} \cdot \vec{\Pi} \\ i\left(\vartheta - \frac{q}{r}\right) + \vec{\sigma} \cdot \vec{\Pi} & 0 \end{pmatrix} \quad (3.3)$$

$$Q_a = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\left(\vartheta - \frac{q}{r}\right) \sigma_a - i\Pi_a + \epsilon_{abc}\Pi_b\sigma_c \\ -\left(\vartheta - \frac{q}{r}\right) \sigma_a + i\Pi_a + \epsilon_{abc}\Pi_b\sigma_c & 0 \end{pmatrix}. \quad (3.4)$$

⁴ $\mathcal{N} = 4$ supersymmetry allows a position-dependent rescaling of the kinetic term [21], but the spectral asymmetry is independent of this deformation, as long as it goes to one at spatial infinity.

which satisfy the algebra (here $m = 1, 2, 3, 4$)

$$\{Q_m, Q_n\} = 2H \delta_{mn} . \quad (3.5)$$

The complete spectrum of this Hamiltonian was analyzed in [23], but unfortunately these authors stopped short of computing the density of states in the continuum. We shall revisit this computation, adapting the classic treatment of the electron-monopole system without potential in [24].

The Hamiltonian (3.1) commutes with the total angular momentum operator [24]

$$\vec{J} = \vec{r} \wedge (\vec{p} - q\vec{A}) - q \frac{\vec{r}}{r} + \frac{1}{4} \vec{\sigma} \otimes (1_2 - \sigma_3), \quad [J_a, J_b] = i\epsilon_{abc} J_c. \quad (3.6)$$

The Schrödinger equation $H\Psi = E\Psi$ can be solved by separating the angular and radial dependence. For this we diagonalize J_3 and \vec{J}^2 denoting by m and $j(j+1)$ their eigenvalues. For the spin 0 part (corresponding to the first two entries of the eigenvector Ψ), we write

$$\Psi = f(r) Y_{q,j,m} , \quad j \geq |q|, \quad j - q \in \mathbb{Z}, \quad (3.7)$$

where $Y_{q,l,m}$ are the monopole harmonics (also known as spin-weighted spherical harmonics), given in the patch around $\theta = 0$ by [25]

$$Y_{q,l,m} = 2^m \sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-q)!(l+q)!}} (1 - \cos\theta)^{-\frac{q+m}{2}} (1 + \cos\theta)^{\frac{q-m}{2}} P_{l+m}^{-q-m, q-m}(\cos\theta) e^{i(m+q)\phi}. \quad (3.8)$$

Here $P_n^{\alpha,\beta}(x)$ are the Legendre polynomials

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{\alpha+n} (1+x)^{\beta+n}]. \quad (3.9)$$

Using

$$(\vec{p} - q\vec{A})^2 = -\frac{1}{r} \partial_r^2 r + \frac{1}{r^2} \left(\vec{r} \wedge (\vec{p} - q\vec{A}) \right)^2 \quad (3.10)$$

$$\left(\vec{r} \wedge (\vec{p} - q\vec{A}) \right)^2 = \vec{L}^2 - q^2 = -\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta - \frac{1}{\sin^2\theta} (\partial_\phi + iq(\cos\theta - 1))^2, \quad (3.11)$$

one can show that the radial wave function satisfies [25]

$$\left[-\frac{1}{2m} \frac{1}{r} \partial_r^2 r + \frac{\nu^2 - q^2 - \frac{1}{4}}{2mr^2} + \frac{1}{2m} \left(\nu - \frac{q}{r} \right)^2 - E \right] f(r) = 0, \quad (3.12)$$

where $\nu = j + \frac{1}{2}$.

For the spin 1/2 part (corresponding to the last two entries of the eigenvector Ψ), the angular dependence is a linear combination of modes with orbital momentum $j - \frac{1}{2}$ and $j + \frac{1}{2}$ [24],

$$\phi_{j,m}^{(1)} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}} Y_{q,j-\frac{1}{2},m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}} Y_{q,j-\frac{1}{2},m+\frac{1}{2}} \end{pmatrix}, \quad \phi_{j,m}^{(2)} = \begin{pmatrix} -\sqrt{\frac{j-m+1}{2j+2}} Y_{q,j+\frac{1}{2},m-\frac{1}{2}} \\ \sqrt{\frac{j+m+1}{2j+2}} Y_{q,j+\frac{1}{2},m+\frac{1}{2}} \end{pmatrix}, \quad j - q \in \mathbb{Z} + \frac{1}{2} \quad (3.13)$$

The first set of modes occurs for $j \geq |q| + \frac{1}{2}$ while the second occurs for $j \geq |q| - \frac{1}{2}$. In order to diagonalize the action of $\vec{\sigma} \cdot \vec{r}$ and $\vec{\sigma} \cdot (\vec{p} - q\vec{A})$, which commute with \vec{J} , it is convenient to introduce the linear combinations (for $j \geq |q| + \frac{1}{2}$) [24]

$$\xi_{j,m}^{(+)} = c_+ \phi_{j,m}^{(1)} - c_- \phi_{j,m}^{(2)}, \quad \xi_{j,m}^{(-)} = c_- \phi_{j,m}^{(1)} + c_+ \phi_{j,m}^{(2)} \quad (3.14)$$

where the coefficients

$$c_{\pm} = \frac{q (\sqrt{2j+1+2q} \pm \sqrt{2j+1-2q})}{|q| \sqrt{2(4j+2)}} \quad (3.15)$$

satisfy $c_+^2 + c_-^2 = 1$. Using $\vec{r} \cdot \vec{\sigma} = 2r \sqrt{\frac{\pi}{3}} \begin{pmatrix} Y_{0,1,0} & \sqrt{2}Y_{0,1,-1} \\ -\sqrt{2}Y_{0,1,1} & -Y_{0,1,0} \end{pmatrix}$, and the multiplication rule

$$\begin{aligned} Y_{q_1,j_1,m_1} Y_{q_2,j_2,m_2} &= \sum_{j_3=\max(|j_1-j_2|, |m_1+m_2|)}^{j_1+j_2} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}} \langle j_1, -q_1, j_2, -q_2 | j_3, -q_1 - q_2 \rangle \\ &\times \langle j_1, m_1, j_2, m_2 | j_3, m_1 + m_2 \rangle Y_{q_1+q_2, j_3, m_1+m_2} \end{aligned} \quad (3.16)$$

for monopole harmonics, one can show that these combinations satisfy, for any $f(r)$,

$$\begin{aligned} (\vec{\sigma} \cdot \vec{r}) f(r) \xi_{j,m}^{(\pm)} &= -r f(r) \xi_{j,m}^{(\mp)}, \\ \vec{\sigma} \cdot (\vec{p} - q\vec{A}) f(r) \xi_{j,m}^{(\pm)} &= i (\partial_r + r^{-1}(1 \mp \mu)) f(r) \xi_{j,m}^{(\mp)}, \end{aligned} \quad (3.17)$$

where we defined

$$\mu = \sqrt{(j + \frac{1}{2})^2 - q^2}. \quad (3.18)$$

Using the fact that the Hamiltonian in the spin 1/2 sector can be written as

$$H_{1/2} = \frac{1}{2m} \left(\vec{\sigma} \cdot (\vec{p} - q\vec{A}) \right)^2 - \frac{q}{2m} \vec{B} \cdot \vec{\sigma} + \frac{1}{2m} \left(\vartheta - \frac{q}{r} \right)^2 \quad (3.19)$$

and the identity

$$\left(\partial_r + \frac{1 \pm \mu}{r} \right) \left(\partial_r + \frac{1 \mp \mu}{r} \right) = \frac{1}{r} \partial_r^2 r - \frac{\mu(\mu \mp 1)}{r^2}, \quad (3.20)$$

we find that its action on $f(r) \xi_{j,m}^{(\pm)}$ is given by

$$\begin{aligned} H_{1/2} \cdot f(r) \begin{pmatrix} \xi_{j,m}^{(+)} \\ \xi_{j,m}^{(-)} \end{pmatrix} &= \left[-\frac{1}{2m} \frac{1}{r} \partial_r^2 r + \frac{\mu^2}{2mr^2} + \frac{1}{2m} \left(\vartheta - \frac{q}{r} \right)^2 \right] \cdot f(r) \begin{pmatrix} \xi_{j,m}^{(+)} \\ \xi_{j,m}^{(-)} \end{pmatrix} \\ &+ \frac{1}{2mr^2} \begin{pmatrix} -\mu & q \\ q & \mu \end{pmatrix} \cdot f(r) \begin{pmatrix} \xi_{j,m}^{(+)} \\ \xi_{j,m}^{(-)} \end{pmatrix}. \end{aligned} \quad (3.21)$$

The 2×2 matrix on the second line has eigenvalues $\pm \sqrt{\mu^2 + q^2} = \pm(j + \frac{1}{2})$, and eigenvectors

$$\tilde{\xi}_{j,m}^{(\pm)} = (\mu \mp \sqrt{\mu^2 + q^2}) \xi_{j,m}^{(+)} - q \xi_{j,m}^{(-)} \quad (3.22)$$

Noting that the coefficient of the centrifugal $1/r^2$ term in the potential is proportional to

$$\mu^2 \pm \sqrt{\mu^2 + q^2} = \left(j + \frac{1}{2} \pm \frac{1}{2}\right)^2 - q^2 - \frac{1}{4}, \quad (3.23)$$

we find that the radial wavefunctions $f_{\pm}(r)$ associated to the eigenmodes $\tilde{\xi}_{j,m}^{(\pm)}$ satisfy the same equation as (3.12) with $\nu = j+1$ (for the $+$ sign, which we refer to as the helicity $h = \frac{1}{2}$ mode) or $\nu = j$ (for the $-$ sign, which we refer to as the helicity $- \frac{1}{2}$ mode).

Finally, for $j = |q| - 1/2$, the space of eigenmodes of J^2, J_3 is one-dimensional, spanned by

$$\eta_m \equiv \phi_{j,m}^{(2)} \propto (1 + \cos \theta)^{\frac{j-m}{2}} (1 - \cos \theta)^{\frac{j+m-1}{2}} e^{i(m+j)\phi} \begin{pmatrix} \sin \theta \\ e^{i\phi}(1 - \cos \theta) \end{pmatrix}. \quad (3.24)$$

One has, in place of (3.17),

$$(\vec{\sigma} \cdot \vec{r}) \eta_m = r \frac{q}{|q|} \eta_m, \quad \vec{\sigma} \cdot (\vec{p} - q\vec{A}) f(r) \eta_m = -i \frac{q}{|q|} (\partial_r + r^{-1}) f(r) \eta_m, \quad (3.25)$$

leading to the same equation (3.12) with $\nu = j$.

In summary, the radial equation is given by (3.12) with

$$\nu = j + h + \frac{1}{2}, \quad j = |q| + h + \ell \quad (3.26)$$

with $h = 0$ for the two bosonic degrees of freedom and $h = \pm \frac{1}{2}$ for the spin $1/2$ degree of freedom, and $\ell \in \mathbb{N}$ in all cases. Solutions to (3.12) with energy $E \equiv \frac{k^2}{2m} > \frac{\vartheta^2}{2m}$ are linear combinations⁵

$$r f(r) = \beta M_{-\frac{iq\vartheta}{\sqrt{k^2 - \vartheta^2}}, \nu} \left(2ir\sqrt{k^2 - \vartheta^2}\right) + \gamma W_{-\frac{iq\vartheta}{\sqrt{k^2 - \vartheta^2}}, \nu} \left(2ir\sqrt{k^2 - \vartheta^2}\right), \quad (3.27)$$

where $M_{\lambda,\nu}(z)$ and $W_{\lambda,\nu}(z)$ are Whittaker functions, which are solutions of the second order differential equation

$$\mathcal{D}_{\lambda,\nu} \cdot w(z) \equiv \left[\partial_z^2 - \frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \nu^2}{z^2} \right] w(z) = 0 \quad (3.28)$$

satisfying

$$M_{\lambda,\nu}(z) \underset{z \rightarrow 0}{\sim} z^{\nu + \frac{1}{2}}, \quad W_{\lambda,\nu}(z) \underset{|z| \rightarrow \infty}{\sim} z^{\lambda} e^{-z/2}. \quad (3.29)$$

The solution proportional to M is regular at $r = 0$, while the solution proportional to W describes an outgoing spherical wave. Since

$$M_{\lambda,\nu}(z) = \frac{\Gamma(2\nu + 1)}{\Gamma(\nu - \lambda + \frac{1}{2})} e^{i\pi\lambda} W_{-\lambda,\nu}(e^{i\pi}z) + \frac{\Gamma(2\nu + 1)}{\Gamma(\nu + \lambda + \frac{1}{2})} e^{i\pi(\lambda - \nu - \frac{1}{2})} W_{\lambda,\nu}(z), \quad (3.30)$$

⁵It helps to note that (3.12) is isomorphic to the Schrödinger equation of the hydrogen atom, whose radial wave-functions are linear combinations of $M_{i\eta, \ell + \frac{1}{2}}(2ikr)$ and $W_{i\eta, \ell + \frac{1}{2}}(2ikr)$ where $\eta = q_1 q_2 m/k$, $E = k^2/2m$. (see e.g. [26, Chap. 14.6]).

we find that the S-matrix in angular momentum channel ℓ and helicity h is

$$S_{h,\ell}(k) = \frac{\Gamma(\nu + \lambda + \frac{1}{2})}{\Gamma(\nu - \lambda + \frac{1}{2})} = \frac{\Gamma\left(|q| + \ell + 2h + 1 - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)}{\Gamma\left(|q| + \ell + 2h + 1 + i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)}. \quad (3.31)$$

In particular, bound states correspond to poles of the S-matrix, and occur only when $q\vartheta > 0$, with energy

$$E_{h,\ell,n} = \frac{\vartheta^2}{2m} \left(1 - \frac{q^2}{(|q| + \ell + 2h + n + 1)^2}\right), \quad (3.32)$$

with $n \geq 0$. The energy depends only on the sum $N = \ell + n + 2h$, so the spectrum has additional degeneracies beyond those predicted by rotational symmetry and supersymmetry [19, 23]. The supersymmetric ground state occurs in the $h = -\frac{1}{2}$ sector with $n = \ell = 0$ and has degeneracy $2|q| = |\langle\gamma_1, \gamma_2\rangle|$. Its wave function is $\begin{pmatrix} 0 \\ r^{q-1} e^{-\vartheta r} \eta_m \end{pmatrix}$, in agreement with [20, 4.16].

The density of states (minus the density of states for a free particle in \mathbb{R}^3) is the derivative of the scattering phase, $\rho(k) dk = \frac{1}{2\pi i} d \log S(k)$. The canonical partition function for states of helicity h is therefore

$$\begin{aligned} \text{Tr}_h e^{-2\pi R H} &= \Theta(q\vartheta) \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} (2\ell + 2|q| + 2h + 1) e^{-2\pi R E_{h,\ell,n}} \\ &+ \sum_{\ell=0}^{\infty} (2\ell + 2|q| + 2h + 1) \int_{k=|\vartheta|}^{\infty} \frac{dk \partial_k}{2\pi i} \log \frac{\Gamma\left(|q| + \ell + 2h + 1 - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)}{\Gamma\left(|q| + \ell + 2h + 1 + i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}\right)} e^{-\frac{\pi R k^2}{m}}, \end{aligned} \quad (3.33)$$

where the first term, corresponding to discrete bound states, contributes only when $q\vartheta > 0$. Summing over all types h weighted by fermionic parity $(-1)^{2h}$, only the BPS bound state with $E = 0$ contributes from the first term, while the contribution of the continuum of scattering states simplifies to

$$\sum_{\ell=0}^{\infty} \int_{k=|\vartheta|}^{\infty} \frac{dk \partial_k}{2\pi i} \left[(2\ell + 2|q|) \log \frac{z_\ell}{\bar{z}_\ell} - (2\ell + 2|q| + 2) \log \frac{z_{\ell+1}}{\bar{z}_{\ell+1}} \right] e^{-\frac{\pi R k^2}{m}}, \quad (3.34)$$

where

$$z_\ell = |q| + \ell - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}. \quad (3.35)$$

Cancelling the terms in the sum, only the contribution $\ell = 0$ remains, leading to

$$2|q| \int_{k=|\vartheta|}^{\infty} \frac{dk \partial_k}{2\pi i} \log \left[\frac{|q| - i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}}{|q| + i\frac{q\vartheta}{\sqrt{k^2 - \vartheta^2}}} \right] e^{-\frac{\pi R k^2}{m}} = \frac{2q\vartheta}{\pi} \int_{k=|\vartheta|}^{\infty} \frac{dk}{k\sqrt{k^2 - \vartheta^2}} e^{-\frac{\pi R k^2}{m}}. \quad (3.36)$$

This is in fact the standard spectral asymmetry predicted by Callias' theorem [27]⁶. Indeed, the result (3.36) does not depend on the details of the S-matrix, but only on the ra-

⁶While the Callias theorem is valid a smooth monopole background, the extension to singular monopoles was worked out in [28], and leads to the same spectral asymmetry. I thank A. Royston and D. van den Bleeken for discussions on this matter.

tio $S_{h,\ell+1}(k)/S_{h,\ell}(k)$, which as we discuss in an Appendix, is fixed by supersymmetry in the asymptotic region.

Using

$$\int_{k=|\vartheta|}^{\infty} \frac{dk}{k\sqrt{k^2 - \vartheta^2}} e^{-\frac{\pi R k^2}{m}} = \frac{\pi}{2|\vartheta|} \text{Erfc} \left(|\vartheta| \sqrt{\frac{\pi R}{m}} \right), \quad (3.37)$$

and adding in the bound state contribution, we get, for arbitrary signs of q and ϑ ,

$$\begin{aligned} \text{Tr}(-1)^F e^{-\pi t H} &= -|2q| \Theta(q\vartheta) + \text{sign}(q\vartheta) |q| \text{Erfc} \left(|\vartheta| \sqrt{\frac{\pi R}{m}} \right) \\ &= -q \left[\text{sign}(q) + \text{Erf} \left(\vartheta \sqrt{\frac{\pi R}{m}} \right) \right] \end{aligned} \quad (3.38)$$

This is indeed a smooth function of ϑ , which interpolates from 0 at $\vartheta = -\infty$ to $-2q$ at $\vartheta = +\infty$ when $q > 0$, or from $-2|q|$ at $\vartheta = -\infty$ to 0 at $\vartheta = +\infty$ when $q < 0$ (see Figure 1, which displays the case $q = 1$). Including the degeneracy $\bar{\Omega}(\gamma)\bar{\Omega}(\gamma')$ of the internal degrees of freedom, and the contribution of the center of motion degrees of freedom, given in the last line of (2.12), we find perfect agreement with the two-particle contribution to the index predicted by the formula (1.2).

4. Discussion

In this note, I have shown that the general formula for the index (1.2) in $\mathcal{N} = 2$, $D = 4$ gauge theories correctly reproduces the contribution of the continuum of two-particle states, in the vicinity of a wall of marginal stability where the constituents can be treated as non-relativistic BPS particles. In particular, I demonstrated that the contributions of the BPS bound states and of the two-particle continuum add up to a smooth function of the moduli, even though each of them is separately discontinuous across the wall. This analysis provides a physical justification for the replacement $\text{sgn}(x) \rightarrow \text{Erf}(x)$, which has been postulated in studies of black hole partition functions in order to enforce S-duality or modular invariance [29, 30, 31, 32], a trick borrowed from the mathematics literature on indefinite theta series [33]. It would be very interesting to calculate the contribution of the continuum of multi-particle states away from the wall, a challenge that will require to understand the dynamics of a collection of relativistic mutually non-local particles beyond the BPS regime.

It is worthwhile noting that similar smooth interpolations across walls of marginal stability have been encountered recently in localization computations of the index in gauged supersymmetric quantum mechanics in certain scaling limits [34, 35]. More generally, error function profiles are ubiquitous in the context of Stokes phenomenon [36], which is formally similar with wall-crossing [37]. It would be interesting to explore these connections.

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A. Robustness of the spectral asymmetry

In order to elucidate the origin of the cancellations in (3.34), we need to understand how supersymmetry relates the density of states in the bosonic and fermionic sectors. For this purpose, notice that the operators

$$\begin{aligned} Q_r &= \partial_z - \left[\frac{\nu + \frac{1}{2}}{z} - \frac{\lambda}{2\nu + 1} \right] = \frac{1}{2i\sqrt{k^2 - \vartheta^2}} \left[\partial_r - \frac{\nu + \frac{1}{2}}{r} + \frac{q\vartheta}{\nu + \frac{1}{2}} \right] \\ Q'_r &= \partial_z + \left[\frac{\nu - \frac{1}{2}}{z} - \frac{\lambda}{2\nu - 1} \right] = \frac{1}{2i\sqrt{k^2 - \vartheta^2}} \left[\partial_r + \frac{\nu - \frac{1}{2}}{r} - \frac{q\vartheta}{\nu - \frac{1}{2}} \right] \end{aligned} \quad (\text{A.1})$$

maps solutions of the Whittaker equation (3.28) with parameters (λ, ν) to solutions of the same equation with parameters $(\lambda, \nu + 1)$ and $(\lambda, \nu - 1)$,

$$Q_r \cdot \mathcal{D}_{\lambda, \nu} = \mathcal{D}_{\lambda, \nu+1} \cdot Q_r, \quad Q'_r \cdot \mathcal{D}_{\lambda, \nu} = \mathcal{D}_{\lambda, \nu-1} \cdot Q'_r. \quad (\text{A.2})$$

In fact, Q_r and Q'_r can be interpreted as the supercharge for the radial problem. To see this, consider acting with $\sqrt{2m} Q_{1/2} = \vec{\sigma} \cdot \vec{\Pi} + i(\frac{q}{r} - \vartheta)$ on fermionic eigenfunctions $f_{\pm} \tilde{\xi}_{j,m}^{(\pm)}$: this should produce linear combinations of bosonic eigenfunctions with the same energy and spin $j \pm \frac{1}{2}$, namely $f_1 \phi_{j,m}^{(1)}$ and $f_2 \phi_{j,m}^{(2)}$. Here, f_+, f_-, f_1, f_2 are solutions of the radial equation (3.12) with $\nu = j + 1, j, j, j + 1$, respectively. Indeed, we find

$$\begin{aligned} i\sqrt{2m} Q_4 \cdot f_+ \tilde{\xi}_{j,m}^{(+)} &= -(2j + 1) c_- \left(\partial_r + \frac{j + \frac{3}{2}}{r} - \frac{q\vartheta}{j + \frac{1}{2}} \right) f_+ \phi_{j,m}^{(1)} + 2\mu\vartheta c_- f_+ \phi_{j,m}^{(2)} \\ i\sqrt{2m} Q_4 \cdot f_- \tilde{\xi}_{j,m}^{(-)} &= -(2j + 1) c_+ \left(\partial_r - \frac{j - \frac{1}{2}}{r} + \frac{q\vartheta}{j + \frac{1}{2}} \right) f_- \phi_{j,m}^{(2)} - 2\mu\vartheta c_+ f_- \phi_{j,m}^{(1)} \end{aligned} \quad (\text{A.3})$$

The differential operator in brackets coincides with $r^{-1} \cdot Q'_r \cdot r$ and $r^{-1} \cdot Q_r \cdot r$, up to overall normalization. Acting on the Whittaker wave-functions, we have, as a consequence of (A.2) and (3.29)⁷,

$$\begin{aligned} Q_r \cdot W_{\lambda, \nu}(z) &= \frac{\lambda - \nu - \frac{1}{2}}{2\nu + 1} W_{\lambda, \nu+1}(z) \\ Q_r \cdot W_{-\lambda, \nu}(-z) &= \frac{\lambda + \nu + \frac{1}{2}}{2\nu + 1} W_{-\lambda, \nu+1}(-z) \\ Q_r \cdot M_{\lambda, \nu}(z) &= \frac{(\nu + \frac{1}{2})^2 - \lambda^2}{2(2\nu + 1)^2(\nu + 1)} M_{\lambda, \nu+1}(z) \end{aligned} \quad (\text{A.4})$$

It follows from these relations that the reflection coefficients $A(\lambda, \nu)$, $B(\lambda, \nu)$ defined by

$$M_{\lambda, \nu}(z) = A(\lambda, \nu) W_{-\lambda, \nu}(-z) + B(\lambda, \nu) e^{i\pi(-\nu - \frac{1}{2})} W_{\lambda, \nu}(z) \quad (\text{A.5})$$

satisfy

$$\frac{A(\lambda, \nu + 1)}{A(\lambda, \nu)} = \frac{2(\nu + 1)(2\nu + 1)}{\nu - \lambda + \frac{1}{2}}, \quad \frac{B(\lambda, \nu + 1)}{B(\lambda, \nu)} = \frac{2(\nu + 1)(2\nu + 1)}{\nu + \lambda + \frac{1}{2}}. \quad (\text{A.6})$$

⁷The last equation in (A.4) was noted in [23, VI.24].

Denoting the reflection coefficient $S_\nu(\lambda) = A(\lambda, \nu)/B(\lambda, \nu)$, one has

$$\frac{S_{\nu+1}(\lambda)}{S_\nu(\lambda)} = \frac{\nu + \lambda + \frac{1}{2}}{\nu - \lambda + \frac{1}{2}} \quad (\text{A.7})$$

This relation only depends on the first two equations in (A.4), which in turn follow directly from the action of $Q_r \sim \partial_r + \frac{\lambda}{2\nu+1}$ on the leading asymptotic behavior $W_{\lambda,\nu} \sim z^\lambda e^{-z/2}$ of the incoming/outgoing plane waves. More generally, the ratio (A.7) depends only on the supercharge at radial infinity and should be unaffected by short-distance corrections to the potential or to the conformal factor in the metric.

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